Engineering Notes

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Eigenvalue Properties of Structural Mean-Axis Systems

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I. Introduction

ONE of the most commonly used body axis systems in the analysis of the dynamics of elastic vehicles is the "mean-axis system." The primary motivation for the utilization of this particular axis system is that it engenders a significant simplification in the vehicle equations of motion and decouples structural dynamics from rigid motions. In the course of performing static and dynamic aeroelastic analyses, it is necessary to generate a structural influence coefficient matrix that allows one to compute vehicle deformations relative to the mean-axis system and simultaneously, enforce the mean-axis constraint. Milne[1] showed that the structural influence matrix can be defined as an "obliqud pseudoinverse" of the free-body stiffness matrix, $[K]$. In the sequel, we review the salient properties of Milne's development and then demonstrate some remarkable spectral invariance properties of the mean-axis structural dynamics equations.

II. Summary of Applicable Pseudoinverse Properties

Let T be a linear transformation from the finite dimensional inner product space V into the finite dimensional inner product space W. V and W are defined over the field of complex numbers and the inner product is denoted by $\langle \cdot, \cdot \rangle$. Denote by $N(T)$ the set of $v \in V$ satisfying $Tv = \theta$ and by $R(T)$ the set of $w \in W$ such that $w = Tv$ for some $v \in V$. Denote by $N(T)^\perp$ the set of $v \in V$ such that $\langle v, v_0 \rangle = 0$ for all $v_0 \in N(T)$. Then $V = N(T) \oplus N(T)^\perp$ and $W = R(T) \oplus R(T)^\perp$. The direct sum (denoted by $\oplus$) of subspaces indicated above implies that if $v \in V$, then there exist unique $v_1 \in N(T)$, $v_2 \in N(T)^\perp$ such that $v = v_1 + v_2$. In general, e are infinitely many pairs of subspaces which provide a $\oplus$ decomposition for a given space.

The linear transformation, T^+, which maps W into V is said to be the *pseudoinverse* of T if[5] 1) $T^+ \text{o}\, Tv = v$ for all $v \in N(T)^\perp$, and 2) $T^+ w = \theta$ for all $w \in R(T)^\perp$. The linear transformation $T^+ \text{o} T$ maps V by the sequence of transformations $Tv = w \in W$, and $T^+ w = v_1 \in V$. In Ref. 1 Milne broadened the concept of the pseudoinverse by defining a more general decomposition for subspaces V and W than that outlined above. Let H be a subspace that is isomorphic to $N(T)$; i.e., if $v \in N(T)$, there corresponds one and only onf $\bar{v} \in H$ and conversely, if $\bar{v} \in H$, there corresponds one and only one $v \in N(T)$. Represent the spaces V and W as $V = H^\perp \oplus N(T)$, $W = R(T) \oplus J$, where J is arbitrary of W.

The *oblique pseudoinverse,* $T^+_{H,J}$: $W \rightarrow V$ is defined to have the following properties: a) $T^+_{H,J} \text{o}\, Tv = v$ for all $v_1 H^\perp$, and b) $T^+_{H,J} w = \theta$ for all $w \in J$. Note the parallel of these properties in the definition of the oblique pseudo-inverse to properties 1) and 2) in the definition of the pseudoinverse. Milne gives an extensive development of the properties of the oblique pseudoinverse in Ref. 1. He proves the following theorem that is of particular importance in the area of dynamics.

Theorem

Let $P_1: V \rightarrow V$ be the projection of V onto $N(T)$ along $H^\perp$ and let $P_2: W \rightarrow W$ be the projection of W onto J along $R(T)$; then

$$T^+_{H,J} = (I_V - P_1) \text{ o } \tilde{T} \text{ o } (I_W - P_2)$$

where $\tilde{T}$ is a weak generalized inverse, i.e., an inverse satisfying $T_o \tilde{T}_o Tv = Tv$ for all $v \in V$.

This result is useful in the computation of $T^+_{H,J}$. The projection operators can be given the following matrix interpretation: Let v_T denote the dimension of the null space of $[T]$, let $[N_T]$ be a matrix of v_T basis vectors for $N(T)$, and let $[H]$ be a matrix of basis vectors of H; then $P_1 = [N_T]([H]^T[N_T])^{-1}[H]^T$.

III. Computation of a Mean-Axis Influence Coefficient Matrix

Let the unforced free-body equations be

$$[K]\{\delta\} + [M_\delta]\{\ddot{\delta}\} = 0 \tag{1}$$

where $[K]$ is an $n \times n$ matrix with null space (in general) of dimension 6. We desire to generate a structural influence coefficient matrix $[\tilde{C}]$ for the computation of deformations $\{\delta'\}$ relative to the mean axis system satisfying

$$\{\delta'\}^T[M_\delta][\bar{\phi}_\delta] = \{0\} \tag{2}$$

where $[\bar{\phi}_\delta]$ is the rigid body mode shape matrix and characterizes the null space of $[K]$. In the general case $[\bar{\phi}_\delta]$ is of dimension $n \times (n-6)$. We incur no loss in generality if we assume that $[K]$ is defined with respect to a structural coordinate frame located at the mass center of the reference configuration. Observe from Eq. (2) that if H is defined to be the space spanned by the columns of $[M_\delta]\ [\bar{\phi}_\delta]$, then in view of the nonsingularity of $[M_\delta]$, H is isomorphic to $N(K)$. A "self-equilibrating" load vector $\{f\}$ is defined to be a load vector which produces a zero net force and moment vector on the structure. If $\{f\}$ is self-equilibrating, then

$$[\bar{\phi}_\delta]^T\{f\} = 0 \tag{3}$$

From Eq. (3) it follows that self-equilibrating load vectors belong to $N^\perp(K) = R(K)$. Thus the free-body flexibility matrix $[\tilde{C}]$ must satisfy c) $[\tilde{C}]\ [K]\ |\delta'\} = \{\delta'\}\ \{\delta'\} \in H^\perp$ and d) $\tilde{C}\ \{f\} = \{\theta\}\ \{f\} \in H$. Condition (c) implies that if (δ') satisfies the mean axis constraint (2), then $[\tilde{C}]$ is effectively an inverse for $[K]$. Condition (d) implies that if $\{f\}$ is *not* self-equilibrating, then rigid vehicle motions result and no deformations occur. The required matrix is clearly the oblique pseudo-inverse $[K^+_{H,H}]$. This matrix can be computed by utilizing the theorem of Sec. II.

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$$[\tilde{C}] = ([I] - [\bar{\phi}_\delta]([\bar{\phi}_\delta]^T[M_\delta][\bar{\phi}_\delta])^{-1}[\bar{\phi}_\delta]^T[M_\delta])[\tilde{G}]$$
$$([I] - [M_\delta][\bar{\phi}_\delta]^T([\bar{\phi}_\delta]^T[M_\delta][\bar{\phi}_\delta])^{-1}[\bar{\phi}_\delta]^T) \tag{4}$$

where $[\tilde{G}]$ is a weak generalized inverse which is computed via the following identification. Partition

$$[K] = \begin{bmatrix} [K_{11}], & [K_{12}] \\ [K_{12}]^T, & [K_{22}] \end{bmatrix} \tag{5}$$

where $[K_{11}]$ is $(n-6) \times (n-6)$, $[K_{12}]$ is $(n-6)\times 6$, $[K_{22}]$ is 6×6. A physical interpretation of the above partition is that if the structure is clamped to eliminate the last six degrees of freedom, then rigid structure motions are eliminated and $[K_{11}]$ is nonsingular. (Any appropriate set of six nodal degrees of freedom can be chosen; we assume the above configuration for convenience). The weak generalized inverse $[\tilde{G}]$ is

$$[\tilde{K}] = \begin{bmatrix} [K_{11}]^{-1}, & [0] \\ [0], & [0] \end{bmatrix} \tag{6}$$

Observe that $[K]$ $[\tilde{G}]$ $[K]$ can be written as

$$[K][\tilde{G}][K] = \begin{bmatrix} [K_{11}], & [K_{12}] \\ [K_{12}]^T, & [K_{12}]^T[K_{11}]^{-1}[K_{12}] \end{bmatrix} \tag{7}$$

However, the assumption that elimination of the last six degrees of freedom eliminates rigid motions implies $[K_{22}] = [K_{12}]^T[K_{11}]^{-1}$ $[K_{12}]$ and $[\tilde{G}]$ is therefore a weak generalized inverse.

IV. Eigenvalue Invariance Properties

Consider the free-body vibration problem

$$[K]\{\delta\} + [M_\delta]\{\ddot{\delta}\} = \{\theta\} \tag{8}$$

The above system is characterized by six (in general) zero eigenvalues with eigenvectors corresponding to the columns of $[\bar{\phi}_\delta]$. Denote the $(n-6)$ nonzero eigenvalues of the free-body problem as ω_i^2 and $\{x_i\}$ the corresponding eigenvector

Then

$$[K]\{x_i\} - \omega_i^2[M_\delta]\{x_i\} = \{\theta\},\ i=1,\ldots,n-6 \tag{9}$$

Now the free body vibration problem for the mean axis system is[2]

$$\{\delta'\} + ([I] - [P_1])[\tilde{G}][M_\delta]\{\delta'\} = \{\theta\}\{\delta'\}\epsilon H^\perp \tag{10}$$

(in the case where external forces $\{f\} \neq 0$, the right hand side of Eq. (10) is replaced by $[\tilde{C}]\{f\}$). Observe now that the eigenvalue-eigenvector pair, $(\omega_i^2, \{x_i\})$ must satisfy

$$\omega_i^2[\bar{\phi}_\delta]^T[M_\delta]\{x_i\} = \{\theta\} \tag{11}$$

and hence $\{x_i\}\epsilon H^\perp$, $i=1,\ldots,n-6$. Consider next the product

$$([I] - [P_1])[\tilde{G}][M_\delta]\{x_i\} = ([I] - [P_1])[\tilde{G}]([I]$$
$$- [P_2])[M_\delta]\{x_i\} = [\tilde{C}][M_\delta]\{x_i\} \tag{12}$$

since $\{x_i\}$ $\epsilon H^\perp$. However,

$$[\tilde{C}][M_\delta]\{x_i\} = \frac{1}{\omega_i^2}[\tilde{C}][K]\{x_i\} \tag{13}$$

By property (c) $[\tilde{C}]$ $[K]\{x_i\} = \{x_i\}$ and we conclude that

$$([I] - [P_1])[\tilde{G}][M_\delta]\{x_i\} = \frac{1}{\omega_i^2}\{x_i\} \tag{14}$$

and therefore that $(\omega_i^2, \{x_i\})$ is also an eigenvalue-eigenvector pair for the mean-axis system. Note that the space of admissible solutions for the mean-axis free vibration problem is of dimension *n-6*. In a sense this restriction deletes the nonuniqueness arising in the solution of the free-body vibration equations. (If $\{\delta(t)\}$ is a solution for the free-body vibration equations (8), then $\{\delta(t)\} + \{v\}$, $\{v\}\epsilon N(K)$, $\{v\}$ constant, is also a solution).

V. Example

The following example is intended to serve as a simple illustration of the preceding developments. We consider a uniform beam of length L with mass m and constant EI. For convenience we break the beam into four finite elements having 5 nodes and develop consistent stiffness and mass matrices for the free beam using the methods outlined in Prznieniecki[3] (see Fig. 1). By assuming that the applied moments at each node are zero the free body stiffness matrix can be reduced to

$$[K] = \frac{64EI}{L^3}\begin{array}{c} \begin{array}{ccccc} \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 \end{array} \\ \begin{bmatrix} 1.607 & & & & \\ -3.643 & 9.857 & & \text{symmetric} & \\ 2.571 & -9.429 & 13.714 & & \\ -0.643 & 3.857 & -9.429 & 9.857 & \\ 0.107 & -0.643 & 2.571 & -3.643 & 1.607 \end{bmatrix} \end{array} \tag{15}$$

If rigid motion of the beam is eliminated by restricting δ_1 and δ_5 to be zero, and the resulting submatrix of $[K]$ is inverted, then the weak generalized inverse matrix $[\tilde{G}]$ for this case can be written

$$[\tilde{G}] = \frac{L^3}{64EI}\begin{bmatrix} 0 & & & & \\ 0 & .750 & & \text{symmetric} & \\ 0 & .916 & 1.333 & & \\ 0 & .583 & .917 & .750 & \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

It is a simple matter to verify that $[K]$ and $[\tilde{G}]$ satisfy $[K][\tilde{G}][K] = [K]$. The rigid body mode shape matrix in this case is

$$[\bar{\phi}_\delta] = \begin{bmatrix} 1 & -.5L \\ 1 & -.25L \\ 1 & .00 \\ 1 & .25L \\ 1 & .50L \end{bmatrix} \tag{17}$$

Finally, the standard Guyan reduction procedure[4] yields a free-body mass matrix of

$$[M_\delta] = \frac{m}{4\times 420}\begin{bmatrix} 117.424 & & & & \\ 66.138 & 391.102 & & \text{symmetric} & \\ -24.122 & 32.020 & 374.204 & & \\ 7.638 & -16.898 & 32.020 & 391.102 & \\ -2.076 & 7.638 & -24.122 & 66.138 & 117.423 \end{bmatrix} \tag{18}$$

In this particular case, the influence coefficient matrix for the free-body subject to mean-axis constraints is

$$[\tilde{C}] = \frac{L^3}{64EI}\begin{bmatrix} .6060 & & & & \\ -.1025 & .0354 & & \text{symmetric} & \\ -.3006 & .0229 & .1994 & & \\ -.0195 & -.0116 & .0223 & .0354 & \\ .4595 & -.0195 & -.3006 & -.1025 & 0.6060 \end{bmatrix} \tag{19}$$

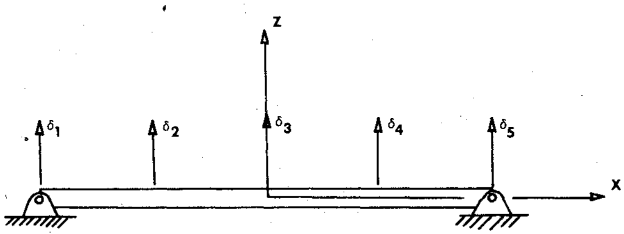

Fig. 1 Definition of nodes for uniform beam.

The finite nonzero engenvalues for both the free-free structural equations and the mean axis structural equations are identical and are $\omega_1^2=0.00469B$, $\omega_2^2=0.3706\text{B}$, and $\omega_3^2=0.17933\text{B}$, where $B=(256\times 420EI)\ /mL^3$. Further the three associated mode shapes are also identical for both structural systems.

A rather extreme example of an incorrect procedure for developing the free-body differential equations can be provided within the context of the above example. Let us suppose that instead of using the appropriate form for $[\tilde{G}]$ given by Eq. (16), we utilize the 3×3 submatrix obtained by omitting the first and fifth rows and columns of Eq. (16). This, in effect, excludes the δ_1 and δ_5 degrees of freedom from the formulation. Similarly, $[\bar{\phi}_\delta]$ of Eq. (17) is modified by deleting the first and fifth rows and a 3×3 mass matrix is obtained by deleting the first and fifth rows and columns of Eq. (18). It is clear that these last two steps significantly alter the mass and mass moment properties of the structure. It turns out that there is only one finite nonzero eigenvalue in this case located at $\omega_1^2=0.02720B$. It is clear that this eigenvalue is significantly in error relative to the results given above. In general the error induced by this procedure is difficult to estimate since it is highly dependent upon magnitude of the mass terms deleted and the location of the clamp points.

VI. Conclusions

We have established that the finite frequencies and mode shapes of the mean axis structural system are identical to the nonzero frequencies and mode shapes of the free structure. The arguments that support this conclusion were based on the use of the oblique matrix pseudo-inverse due to Milne. A simple example was given to illustrate the procedure and to point out some of the difficulties that can arise if the procedure is incorrectly implemented.

References

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